

Inconsistency Reduction in decision making via Multi-objective Optimisation

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Abstract

Within Multi-Criteria Decision Analysis, pairwise comparison facilitates a separation of concerns helping to accurately represent a decision maker's preferences. Inconsistency within a set of pairwise comparisons has adverse effects upon the accuracy of the preferences derived from them. Inconsistency within pairwise comparisons is almost inevitable, hence consideration of its reduction is essential. This paper presents INSITE, an approach to inconsistency reduction within a set of pairwise comparisons via multi-objective optimisation. When seeking to reduce inconsistency within a set of pairwise comparisons there is a trade-off between alteration to the comparisons and the reduction of inconsistency within them. For such trade-offs no trade-off solution is superior per se to the others. Therefore, INSITE seeks to optimally reduce inconsistency within a set of comparisons by modelling inconsistency and alteration as separate objectives. In this way the nature of the trade-offs between inconsistency reduction and alteration are revealed, thus better informing a decision maker's awareness and knowledge of the problem and increasing validity of outcomes by providing a more evidential, transparent, auditable and traceable process. In this way a decision maker can look to make a more informed choice of the level of trade-off that is most suitable for them. INSITE is flexible regarding how inconsistency within judgments is measured; alteration to a decision maker's views is modelled via fine-grained measures of compromise that seek to be meaningful and relevant. Furthermore, the approach allows a decision maker to set constraints on both inconsistency and measures of compromise objectives.

Keywords: Multi criteria analysis, Pairwise comparisons, Inconsistency, Multi-objective optimization; Analytic Hierarchy Process

1 Introduction

Multi-Criteria Decision Analysis (MCDA) seeks to determine the suitability of alternatives of a decision with respect to multiple criteria. The concept of Pairwise Comparison (PC) is employed within many MCDA methods (Hwang & Yoon, 1981; Saaty, 1980, 2001); PC enables decomposition of a larger decision problem into more manageable smaller chunks, facilitating a separation of concerns that ensures an accurate extraction of the preferences of a Decision Maker (DM). For a set of elements under consideration, a PC judgment can be made for each pair of elements and from this set of comparison judgments a one-dimensional ranking of the elements, a *Preference Vector*, can be derived representing a ranking of the set of elements under consideration by the DM.

DMs are subject to fragilities, such as biases, inconsistencies and irrationalities, in their views (French, Maule, & Papamichail, 2009). Within a set of PCs, the unification of the smaller chunks of each PC judgment may result in inconsistency being present in the set of judgments as a whole. Inconsistency within PC used for consideration of more than a handful of elements is almost inevitable (Choo & Wedley, 2004). When inconsistency is present in a set of judgments, any preference vector derived will only be an estimate of the judgments' information. Therefore, inconsistency within a set of judgments can adversely affect the accuracy of a resulting ranking, so consideration of its reduction is important in deriving a more accurate preference vector from a DM's views. Furthermore, tackling the problem of inconsistency with emphasis upon traceability should facilitate greater validity of outcomes for a DM by providing more explanation and greater clarity of the decision process.

In this paper we present **INSITE** (reducINg inconSIsTency in dEcision making) - a flexible and traceable approach to reducing inconsistency within a set of PC judgments via Multi-objective Optimisation (MOO).

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When seeking to reduce inconsistency within a set of pairwise comparisons there is a trade-off between alteration to the judgments and the reduction of inconsistency within them. Previous approaches to inconsistency reduction within a set of PCs facilitate little traceability and offer no consideration of seeking to reveal to a DM the nature of trade-offs between inconsistency reduction and alteration to his/her judgments. Moreover, they offer a DM no control over how inconsistency is measured, with little consideration of alteration to the judgments in terms of measurements of semantic relevance to the DM. INSITE seeks to optimally reduce inconsistency within a set of DM judgments through modelling inconsistency and alteration as separate objectives via MOO. Within INSITE, measurement of inconsistency is adaptive to different user preferences and scenarios. Alteration to a DM's judgments is modelled via measures of compromise, that seek to be semantically meaningful such that a DM can comprehend and relate to them, thus enhancing the knowledge that can be gleaned from the inconsistency reduction process and hence increase understanding of the decision process. Furthermore, INSITE facilitates the setting of constraints defined by the DM both on inconsistency and alteration objectives.

For such trade-off multi-objective problems, without additional information, no trade-off solution is superior per se to the others (Coello, 2006). Therefore, the philosophy of INSITE is to allow a DM to glean knowledge with regards to the trade-offs. Modelling inconsistency via MOO elucidates, and informs the DM of, the trade-offs between reducing inconsistency and altering DM judgments, thereby enhancing understanding of both the problem and its solution space. In this way, INSITE should facilitate greater validity of outcomes by providing more evidence of what is involved in the inconsistency reduction process.

In tackling the problem of inconsistency reduction within a set of DM judgments INSITE, significantly extends work in (Abel, Mikhailov, & Keane, 2013), by looking to optimally reduce inconsistency within a set of DM judgments by modelling inconsistency and alteration to a DM's judgments as separate objectives via MOO. Specifically, in expanding the preliminary work we make the following contributions:

- Defining the mathematical formulation of the objective model for inconsistency reduction via MOO within INSITE;
- Facilitating the setting of DM-defined constraints both on inconsistency and alteration objectives;
- Presenting various experimentation evaluating the approach by exploring:
 - Explicit comparison and evaluation of INSITE to four other approaches to inconsistency reduction;
 - The use of constraints to aid a DM to interactively explore the objective space and hone in on a specific solution in the objective space;
 - Analysis of the nature of trade-off fronts for different inconsistency measures and exploring the effects of using different measures of compromise;
 - How a DM could utilise multiple measures of compromise simultaneously and/or utilise multiple measures of inconsistency simultaneously.
- Incorporating a larger set of inconsistency measures usable by a DM to measure inconsistency reduction. Furthermore, a more comprehensive and up-to-date literature review is presented, to help frame and clarify how INSITE seeks to innovatively tackle the problem of inconsistency reduction.

The rest of the paper is structured as follows: Section 2 reviews the literature regarding measuring and reducing inconsistency within a set of PCs; INSITE is then outlined in Section 3; examples are discussed in Section 4; finally, conclusions are presented in Section 5.

2 Inconsistency Reduction in PC

PC enables a DM to only consider a pair of decision elements and to determine their preference, and strength of preference, between the pair, with respect to an intangible factor. This segmentation of a larger decision problem is achieved through use of the Law of Comparative Judgment (Thurstone, 1927). This ability to take only a pair of elements of a decision at a time, helps to achieve a separation of concerns for the DM and assists them in achieving a more accurate reflection of their judgments (Saaty, 2008; van Til, Groothuis-Oudshoorn, Lieferink, Dolan, & Goetghebeur, 2014).

Given two elements x and y , we denote that the DM prefers element x to element y with the notation $x \phi y$. Various numerical scales may be utilized to represent the strength of preference; the most widely utilized being the Saaty 1-9 scale (Saaty, 1977), where, for example, if element x is preferred 3 times more than element y , this can be denoted as $x \phi y$ with a preference strength of 3. If neither element is preferred over the other, the elements are said to be equally preferred, denoted by $x \sim y$ and the preference strength is represented with the

value 1. Various other scales of differing preference strength intervals have been proposed in the literature, see (Harker & Vargas, 1987; Ishizaka, Balkenborg, & Kaplan, 2011; Lootsma, 1989).

The set of PCs, one for each pairing of elements in a set of elements, along with the self-comparison values and the reciprocal values, can be collated into a two-dimensional Pairwise Comparison Matrix (PCM), as shown for PCM P in (1) for a set of n elements, where a_{ij} represents a PC between elements i and j .

$$P = \begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ 1/a_{12} & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \dots & 1 \end{pmatrix} \quad (1)$$

For a completed PCM of the type (1) of $(n \times n)$ elements, there exists a preference vector $w = [w_1, w_2, \dots, w_n]^T$, where w_i represents the weighting of the element i for $i = 1$ to n . A preference vector ranking of the n elements can be derived through the use of a *Prioritization Method*. Many Prioritization Methods exist for this task; see (Choo & Wedley, 2004) for a comprehensive discussion.

2.1 Inconsistency

The consistency of a PCM is the extent to which its set of judgments are coherent. When there is inconsistency present in a PCM, any preference vector derived from it will only be an estimate of its implicit ranking information (Choo & Wedley, 2004). Consequently, different prioritization methods may derive different preference vector estimates. The greater the amount of inconsistency present, the less accurately the derived preference vector represents the PCM's judgment information. Approximations of highly inconsistent PCMs produce large errors, hence "*approximations from such matrixes make little practical sense*" (Koczkodaj & Szarek, 2010). Inconsistency within a PCM of more than a handful of elements is almost inevitable (Choo & Wedley, 2004) and therefore needs to be considered.

Inconsistency within a set of PC judgments may be categorized as either ordinal or cardinal, both being important considerations for a DM. Ordinal inconsistency identifies inconsistent information without the strengths of preference of the DM's judgments being considered. For example, given a set of 3 elements, x , y and z : if $x > y$, $y > z$ and $z > x$, then the judgments are intransitive and contradictory, and ordinal inconsistency is present. Ordinal inconsistency can also be present within a set of judgments containing equal preference judgments. For example, if x and y are equally preferred ($x \sim y$) then for the set of judgments to be ordinally consistent the remaining judgments must be: $x > z$, and $y > z$, or $x < z$, $y < z$, or $x \sim z$, $y \sim z$. Cardinal inconsistency identifies inconsistency between a set of judgments taking into account the strength of preference of each judgment. For a set of judgments to be cardinally consistent then each judgment j should maintain transitivity - that is, the relation between a first element and a second and between a second element and a third should hold between the first and third. For example, considering a set of 3 elements x , y and z : if $x > y$ with a preference strength of a and $y > z$ with a preference strength of b , then, for the judgment set to be cardinally consistent, the final judgment between elements x and z would need to be such that $x > z$ with a preference strength of $a*b$.

Next we discuss various measures that quantify the level of inconsistency within a set of judgments.

2.1.1 Inconsistency measures

A number of ways to measure inconsistency within a set of PC judgments have been proposed. INSITE implements multiple inconsistency measures so as to be adaptive to different DM preferences and scenarios.

2.1.1.1 3-Way Cycles

The number of 3-way cycles present within a PCM is an ordinal measure of inconsistency. A measure proposed in (Gass, 1998) formulates the problem as a tournament ranking (with 0 and 1 utilised to represent judgments as losses and wins respectively), without consideration of preference equivalence. Alternatively, the presence of 3-way cycles, including consideration of equal preference judgments, can be determined via an algorithm (Kwiesielewicz & van Uden, 2004). This can be utilized to determine the total number of 3-way cycles within a PCM, usually denoted as L . We only need to consider cycles of 3 elements as it has been shown that eliminating all 3-way cycles ensures elimination of cycles of higher orders (Harary & Moser, 1966).

2.1.1.2 Consistency Ratio

The Consistency Ratio (CR) proposed in (Saaty, 1977) is a measure of the amount of cardinal inconsistency present within a PCM. Firstly, the eigenvalue of the largest eigenvector of the PCM (λ_{max}) is calculated. When an order n PCM is perfectly consistent then $\lambda_{max} = n$. Next, the Inconsistency Index (CI) of the PCM is determined.

$$CI = (\lambda_{max} - n) / (n - 1) \quad (2)$$

The CR is then found by dividing the CI by the Random Consistency Index (RI) for the order of the PCM. The RI values represent the average inconsistency found over 50,000 trials of randomly generated matrixes for each PCM order, see (Saaty, 1980).

$$CR = CI / RI \quad (3)$$

The lower the CR value, the lower the amount of cardinal inconsistency present in the PCM. Saaty further proposed an acceptability threshold value of a PCM's CR value (Saaty, 1980). The threshold is designed to be an indicator as to whether a PCM is consistent enough for a satisfactory preference vector estimate to be derived. Using this threshold, when a PCM has a CR value of 0.1 or less, it is considered to be acceptable. It has been argued that the choice of the 0.1 threshold to determine an acceptable level of inconsistency is arbitrary and not based upon solid foundations (Koczkodaj, 1993). Therefore, giving a DM control over such a threshold is likely to be beneficial.

2.1.1.3 Consistency Measure

The Consistency Measure (CM), proposed in (Koczkodaj, 1993) and generalised in (Duszak & Koczkodaj, 1994), is a cardinal inconsistency measure based upon the transitive properties of a set of judgments. CM is a more fine-grained alternative of the CR measure that considers the inconsistency between each triple of judgments, to identify the set of triple judgments that are the most inconsistent. Considering each possible set of 3 judgments at a time CM determines the inconsistency of a triple of judgments between elements x , y and z via:

$$CM_{xyz} = \min \left(\frac{|j_{xy} - j_{xz}j_{zy}|}{j_{xy}}, \frac{|j_{xy} - j_{xz}j_{zy}|}{j_{xz}j_{zy}} \right) \quad (4)$$

CM gives not just a measure of inconsistency but also identification of where the highest levels of inconsistency within the set of judgments occurs.

2.1.1.4 Geometric Consistency Index

The Geometric consistency Index (GCI) (Aguarón & Moreno-Jiménez, 2003) is an inconsistency measure based upon the distance between a preference vector derived using the Geometric Mean (GM) prioritization method (Crawford, 1987) and the original judgments. GCI is calculated via:

$$GCI = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^n (\log a_{ij} - \log(\frac{w_i}{w_j}))^2 \quad (5)$$

where w_i is the ranking value for element i in the preference vector derived from the GM prioritization method. Comparison of GCI and CR showed them to have an almost linear relationship (Aguarón & Moreno-Jiménez, 2003). A threshold of acceptability of GCI has also been proposed (Aguarón & Moreno-Jiménez, 2003), when $n=3$ $GCI \leq 0.31$, $n=4$ $GCI \leq 0.35$, when $n>4$ $GCI \leq 0.37$.

2.1.2 Previous approaches to Inconsistency Reduction

There are various ways to tackle inconsistency, such as (1) getting the DM to review their judgments; (2) automatically altering the judgments; and (3) proceeding but attempting to take the inconsistency knowledge into consideration. INSITE is focused upon the second of these. Various approaches have been proposed to seek to automatically reduce inconsistency within a set of judgments.

A convergence algorithm approach was proposed in (Xu & Wei, 1999) which obtains an altered PCM that has a CR measure below a threshold ($CR < 0.1$). The approach can alternatively be applied iteratively to reduce CR to 0. The algorithm looks to find a cardinally consistent altered PCM as a single objective whilst seeking to ensure the amount of departure from the original judgments is below given ranges (via hard constraints). Given that values of a found altered PCM solution are composed of judgment values that fall outside of the original judgment scale, it may be difficult for a DM to comprehend how their judgments have changed. Additionally, the constraints used to measure departure from the original judgments are likely to be difficult for a DM to semantically comprehend and relate to with regard to how their judgments have changed, obfuscating understanding of the process. Furthermore, as alteration is used only as a constraint there is no explicit consideration of minimizing the amount of alteration in pursuit of inconsistency reduction.

A similar convergence algorithm approach was proposed in (Cao, Leung, & Law, 2008). Again, only the cardinal inconsistency measure CR is considered with the aim to find a solution below a threshold ($CR < 0.1$). The values of altered PCMs found are composed of judgment values that fall outside of the original scale utilised, which again hider a DM's comprehension of how their judgments have altered. Alteration is considered (through similar calculations as defined in (Xu & Wei, 1999)) again as hard constraints to determine if the found altered PCMs are feasible. The alteration constraints to measure departure from the original judgments are again likely to be difficult for a DM to relate to with regard to how their judgments have changed.

An approach to reducing ordinal inconsistency is proposed in (Siraj, Mikhailov, & Keane, 2012). This approach seeks to reduce the number of 3-way cycles within a PCM via an iterative process of judgment reversals. At each iteration it seeks to reverse a judgment that will result in the maximum reduction of 3-way cycles to converge to a solution PCM without any 3-way cycles. On each iteration the approach identifies the judgment which will have the most impact upon ordinal inconsistency reduction. Through removing ordinal inconsistency optimally, fewer iterations should be required and thus fewer reversals required to reach a set of fully consistent judgments. If multiple judgments represent the maximum reduction of 3-way cycles then cardinal inconsistency is considered only as a tiebreaker to determine which judgment is reversed. Here the cardinal inconsistency of a judgment is measured via the amount of discrepancy between the judgment's strength and the measurement of indirect judgment strength.

Inconsistency reduction has also been addressed by the approach in (Costa, 2011) which utilizes genetic algorithms. Only cardinal inconsistency (CR) is considered as a single objective to look to find solutions for which cardinal inconsistency is below a threshold, again $CR < 0.1$. The amount of alteration between found solutions and the original judgments is not explicitly considered. Additionally, solutions are modelled in such a way that the reciprocal property of the PCM is not always maintained in found solutions, which may introduce additional inconsistency into the judgments. The amount by which the reciprocal property is violated is defined via a user-set tolerance parameter.

A genetic algorithm is also utilised in (Wang, Liu, & Pang, 2012) to reduce inconsistency of a PCM; here the PCM and the altered PCM are represented as fuzzy numbers. This approach only considers cardinal inconsistency looking to find a solution with a lower CR value. When evaluating solutions during optimisation, the CR of individuals and the alteration of the amount of change are considered as a single objective. Individuals with feasible CR are assigned a high evaluation value. Alteration is then considered only to rank the remaining solutions with CR 0.1 or higher.

Similarly the approach in (Sun, Liu, & Zhang, 2011) looks to find an altered solution with reduced inconsistency considering only cardinal inconsistency. The approach looks to find altered solutions with the lowest value of CR measure. The approach models the problem as a non-linear programming model and a genetic algorithm is utilised to solve it. During the operation of the genetic algorithm individuals are evaluated via the level of cardinal inconsistency and consideration of alteration to the PCM via their combination into a single objective function. The level of CR is used to measure cardinal inconsistency and similarity between the initial PCM and the solution set is measured as a log-based calculation of the amount of change between the two judgments sets (Sun, Liu, & Zhang, 2011). The measure of alteration utilised is difficult for a DM to semantically interpret with respect to the alteration their judgments have undergone.

Ant colony optimization is utilised within an approach to inconsistency reduction (Girsang, Tsai, & Yang, 2015). Only cardinal inconsistency is considered, with the aim to find a solution with CR below 0.1 whilst also considering the amount of departure from the original judgments. These objectives are not considered

separately within MOO, rather the approach seeks to find a solution that satisfies the CR threshold for which the amount of departure from the original judgments is lowest. In this sense, CR is treated as a constraint. Departure of a solution from the initial judgments is measured by the difference index measure, which is hard for a DM to interpret and relate to his/her judgments. Solutions are found within the bounds of the scale utilized, however judgments can fall between the scale's step values hindering comprehension by a DM of how their judgments have been altered.

Inconsistency reduction for a PCM represented as either crisp or fuzzy numbers has been proposed in (Zhang, Sekhari, Ouzrout, & Bouras, 2014). Here inconsistency with respect to cardinal inconsistency is considered and linear programming is utilized to find a solution with lower CR whilst also considering the amount of departure from the original judgments. Inconsistency and alteration to judgments are considered via single objective optimization based on linear algebra. Departure from the original judgments is considered via measurements that are difficult for a DM to semantically interpret, and found solutions may contain judgments outside the original utilised scale.

An approach in (Bozóki, Fülöp, & Poesz, 2015) tackles inconsistency reduction considering multiple measures of inconsistency, however only cardinal measures are considered. In the approach a single measure of alteration, the number of judgments that change, is considered. The approach, using nonlinear mixed-integer optimization, has two modes of operation that both seek to optimise a single objective given a constraint. The approach seeks to determine the minimum number of judgment changes required to reach a threshold value of inconsistency (0.1 when the CR measure is utilized). Alternatively, the approach seeks to determine the minimal level of inconsistency that can be achieved given a constraint of the number of judgments that can change. As the approach only considers single objective optimisation, inconsistency reduction and alteration are not simultaneously optimised in either mode of operation.

Some approaches look to consider both ordinal and cardinal inconsistency, such as the approach in (Li & Ma, 2007), which models a PCM via Gower plots to then determine the ordinal and cardinal inconsistency present. The approach then facilitates a DM to iteratively look to reduce ordinal and/or cardinal inconsistency in his/her judgments. Although the DM has the ability to tackle both ordinal and cardinal inconsistency it is via measures calculated in relation to the Gower plots offering the DM no control in choice of measures. Moreover, the two models, for reducing ordinal and cardinal inconsistency respectively, are optimised with respect to the Gower plot representations and seek solutions without consideration of their interpretability to the DM, making it harder to relate to with respect to his/her original judgments. In addition, there is no consideration of seeking to reveal to the DM the nature of trade-offs between inconsistency reduction and alteration to their judgments.

Kou, Ergu, & Shang (2014) propose a method to tackle cardinal and ordinal inconsistency using an adapted Hadamard model. The approach presents a method which seeks to obtain an altered PCM that has a CR measure below a threshold ($CR < 0.1$). The DM's judgments are transformed into a Hadamard product induced bias matrix (HPIBM). From this, the approach looks to determine the most (cardinally) inconsistent element, as the largest value in this new representation, then determine a more appropriate value, calculated based on information relating to indirect judgments between the set of judgments. If after the adjustment the PCM is still greater than 0.1, then the next most inconsistent judgment is iteratively chosen until the threshold is met. In seeking reduction to a pre-determined threshold there is no consideration of being able to reveal to the DM the nature of trade-offs between inconsistency reduction and alteration to their judgments. The approach can tackle ordinal inconsistency by looking to identify three-way cycles from the HPIBM and then look to eliminate all 3-way cycles by utilising information pertaining to indirect judgments.

In summary, in approaches to alter the judgments in a PCM aiming to reduce inconsistency there is no consideration of seeking to reveal to the DM knowledge of the nature of trade-offs between inconsistency reduction and alteration to their judgments, from which a DM can look to make an informed choice of the level of trade-off that is most suitable for them. Moreover, approaches mostly focus upon either ordinal or cardinal inconsistency, and offer no facilities for a DM to explicitly choose how inconsistency is measured. When both ordinal and cardinal inconsistency are considered the DM cannot explicitly control the inconsistency reduction in terms of what is *most* important to them in terms of the inconsistency measures. Furthermore, when seeking to reduce inconsistency to a threshold value, most approaches offer no control for a DM to define the threshold value. When approaches consider alteration to the judgments in pursuit of inconsistency reduction it is not considered explicitly, but rather as part of a single objective or as a constraint. Additionally, when alteration is considered, little effort is made to provide semantically meaningful measurements to which a DM can relate. Furthermore, approaches offer no control for a DM to choose how alteration is measured to suit their preferences. Some approaches do not always maintain the reciprocal properly of the original PCM, and others find solutions with values outside the original judgment scale. INSITE seeks to address these limitations.

3 INSITE

This section describes INSITE. Firstly, an overview of INSITE and a description of its stages are presented; next, how alteration to judgments in INSITE is modelled is discussed; a formal definition of INSITE is then presented, followed by discussions of its implementation.

3.1 Overview of INSITE

INSITE looks to optimally reduce inconsistency within a set of DM judgments by modelling inconsistency and alteration to a DM's judgments as separate objectives via MOO. Specifically, this paper expands preliminary work by: defining a detailed mathematical formulation of the objective model for MOO inconsistency reduction and facilitating the setting of DM-defined constraints both on inconsistency and alteration objectives. Moreover, more complex examples are presented exploring: Explicit comparison to four other approaches to inconsistency reduction; the use of constraints to aid interactive exploration of the objective space by a DM to hone in on a specific solution; analysis of the nature of trade-offs for different inconsistency measures as well as the effects of using different measures of compromise; the use of multiple measures of compromise and/or utilise multiple measures of inconsistency simultaneously. Finally, a larger set of inconsistency measures is incorporated and usable by a DM to measure inconsistency reduction.

When seeking to reduce inconsistency within a set of pairwise comparisons there is a trade-off between alteration to the judgments and the reduction of inconsistency within them. INSITE takes in a set of judgments from a DM and looks to find *Altered Solutions*, which are new judgment sets that will be derived from the MOO process. From a set of trade-off Altered Solutions for such a multi-objective problem no solution is superior to the others and, (without additional information), all solutions are equally preferred. Both cardinal and/or ordinal inconsistency can be considered, giving a DM control over the type of inconsistency reduction required. INSITE facilitates inconsistency reduction whilst also looking to minimise the amount of alteration to achieve the reduction. Alteration to judgments is explicitly considered in INSITE using measures of compromise, which are discussed in Section 3.2. The use of measures of compromise give a DM control over how alteration is measured to meet their needs. Moreover, they help a DM glean greater understanding of the process and knowledge of the trade-offs involved, thereby enhancing traceability. In this way, a DM can make a more informed choice of the level of trade-off that is most suitable for them.

Additionally, INSITE allows a DM to set constraints relating to the amount of inconsistency reduction they are seeking to achieve as well as the amount of alteration they are willing to tolerate. In INSITE constraints can be utilised in an interactive iterative manner. An initial search can reveal to the DM the nature of the objective space regarding inconsistency reduction for their chosen objectives. Informed by this knowledge, the DM can then set feasible (and achievable) constraints to the problem. Constraints can then be iteratively added to drill down into the objective space to aid the DM in selecting a single solution. Furthermore, INSITE seeks to alter judgments in such a way that the judgments maintain the original scale utilised by the DM during judgment elicitation, allowing a DM to more easily discern how their judgments have altered.

INSITE is independent of a specific prioritization method, so any method can be utilised to derive a ranking from an Altered Solution found, enabling INSITE to be adaptive to different scenarios and DM preferences. INSITE implements the 1-9 scale to elicit judgments (this scale is utilised within our examples) however the approach is independent of a specific scale and could be extended to be used with any bounded scale.

The stages of INSITE, shown in Figure 1, can be summarized as follows:

1. The number of elements of the problem is defined;
2. Judgments are elicited from the DM pertaining to their preferences between the elements;
3. The objectives for the MOO process are selected by the DM consisting of one or more measures of compromise objectives (see Section 3.2) and one or more inconsistency measure objectives (see Section 2.1.1);
4. The set of objectives are then utilised within a MOO framework to find the set of trade-off Altered Solutions between the objectives, see Section 3.3;

5. Analysis of the set of Altered Solutions can then be performed to aid the DM towards selection of a single solution from the set of solutions found. Such analysis can be performed via:
 - i. Gleaning knowledge of the trade-offs between the chosen objectives of solutions in the objective space, and of the nature of the inconsistency reduction and the compromise to facilitate it. Inspection of, and comparisons between, the solutions found can be performed to aid a DM in selection of a final solution. Additionally, such analysis of the nature of the trade-off front for the problem may aid a DM in recalibrating their goals as to what are achievable levels of reduction for various amounts of alteration;
 - ii. Through analysis of the set of found Altered Solutions, a DM can iteratively add feasible constraints to gradually drill down to a sub-region of the objective space to help select a final solution;

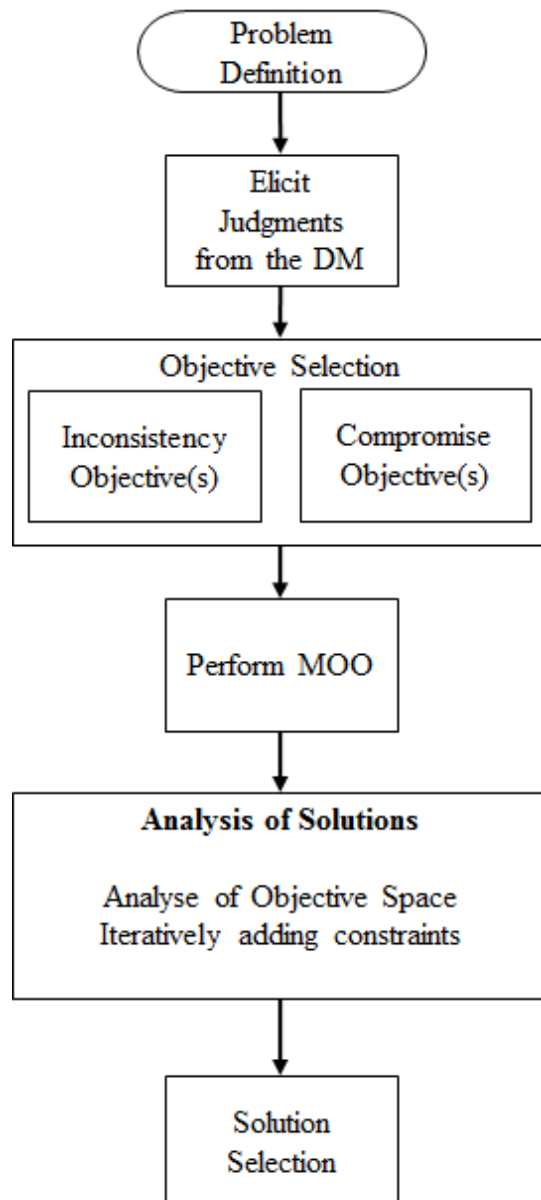


Figure 1: Flowchart of INSITE stages

3.2 Measures of Compromise

During inconsistency reduction a DM's judgments will be altered to reduce the amount of inconsistency present. INSITE utilises the Measures of Compromise (Abel, Mikhailov, & Keane, 2013) to calculate alteration

to a set of judgments in ways that are semantically meaningful and cognate for a DM. Measures that are meaningful to a DM should aid in making inconsistency reduction more understandable, auditable and traceable, and aid a DM in setting constraints that are more informed. The use of the measures of compromise provide a user with choice over how preservation of the original judgment information is to be defined, to thus be most appropriate to him/her, with measures that are semantically meaningful and interpretable to his/her original judgments. In this way INSITE allows a DM to choose between, for example, “fewer judgments changing (perhaps a lot)” or “many judgments changing a little” constitutes better preservation of the original judgment information for them. The measures of compromise are measures between two sets of judgments, whereas measures such as Least Absolute Error (Choo & Wedley, 2004) and Minimum Violations (Golany & Kress, 1993) are generally utilized to refer to measurements between a set of judgments and a derived preference vector. Hence, slightly different names are utilized here to differentiate, looking to emphasise the measures of compromises’ application to judgment set to judgment set comparison.

Given a problem with N elements we elicit PCs from a DM of each pair of elements within the N set of elements to construct a PCM. Due to the reciprocal property and self-comparisons of a set of judgments, the minimum number of judgments J required to construct a complete PCM is $N(N - 1)/2$. From a completed PCM we extract a set of J judgments that will contain all the information to reconstruct the PCM. J is selected as the top triangle of a completed PCM as from this set of judgments the whole PCM can be reconstructed. Therefore, J is the set of judgments from the PCM without self-comparison and reciprocal judgments. J is chosen as the upper triangle of a completed PCM as it will contain all information such that all the self-comparison and reciprocal judgments’ information can be inferred from it. Judgments themselves, from which a completed PCM is constructed, may be elicited from a DM in various ways to suit their preferences; from the completed PCM, the top triangle of judgments is then selected to define J . Given an *Original judgment set* (O) represented as a set of judgments $\{o_1, o_2, \dots, o_j\}$ of cardinality J , we look to measure the amount of alteration between O and a second *Altered judgment set* (A) of judgments $\{a_1, a_2, \dots, a_j\}$, via a measure of compromise. The Measures of Compromise are repeated here for clarity and completeness:

1. **Number of Judgment Violations (NJV):** a measure of the number of the original set of judgments that have changed, where δ evaluates to 0 or 1 for each Boolean evaluation.

$$NJV = \sum_{j=1}^J \delta(o_j \neq a_j) \quad (6)$$

NJV may be useful when a DM is seeking solely to look to minimise the number of their judgments that change in the pursuit of inconsistency reduction.

2. **Total Judgment Deviation (TJD):** a measure of the total amount of change between each judgment from the original judgments and an altered judgment set.

$$TJD = \sum_{j=1}^J |o_j - a_j| \quad (7)$$

TJD may be useful when a DM is seeking to minimize the total amount of steps along the judgment scale their judgments undergo in pursuit of inconsistency reduction.

3. **Squared Total Judgment Deviation (STJD):** a variant of TJD that gives more emphasis to larger amounts of deviation to a judgment.

$$STJD = \sum_{j=1}^J (o_j - a_j)^2 \quad (8)$$

STJD may be useful when a DM is seeking to avoid large judgment changes in the pursuit of inconsistency reduction.

4. **Number of Judgment Reversals (NJR):** a measure of the number of judgments from the original set whose preference has been inverted, with consideration of half reversals. Given an original judgment between elements A and B for which A is preferred to B, $A > B$, (therefore with a value from the 1-9 scale greater than 1). If within the altered judgment set, B is now preferred to A, $A < B$, (therefore with a value from the 1-9 scale of less than 1) then a reversal has occurred. Similarly, if instead for the original judgment between elements A and B, B is preferred to A ($A < B$) and in the altered judgment set A is now preferred to B ($A > B$) then a reversal has occurred. Moreover, if in the original judgment the DM stated equal preference between elements A and B, $A \sim B$, (with a value from the 1-9 scale of 1) and then, within the altered set, either A is now preferred to B or B is now preferred to A (therefore no longer a value of 1) then a half reversal has occurred. Furthermore, if the original judgment between elements A and B are not of equal preference (so either $A > B$ or $A < B$), and then within the altered set the elements are now equally preferred ($A \sim B$), then a half reversal has occurred.

$$NJR = \sum_{j=1}^J R_j \quad (9)$$

where

$$R_j \begin{cases} 1: o_i > 1 \text{ and } a_i > 1 \\ 1: o_i < 1 \text{ and } a_i < 1 \\ 0.5: o_i = 1 \text{ and } a_i \neq 1 \\ 0.5: o_i \neq 1 \text{ and } a_i = 1 \\ 0: \text{otherwise} \end{cases}$$

NJR may be useful for a DM seeking solely to minimise the number of preference changes to their judgments in the pursuit of inconsistency reduction.

3.3 Multi-objective optimisation model

Many real-world problems consist of multiple, frequently conflicting, objectives. Such problems may be tackled by weighting each objective function and then combining them together to create a single objective. However, such an approach requires that the weights of each objective are defined by the DM prior to optimisation and reveal no knowledge regarding the relationship and nature of conflict between the objectives. Alternatively, the objectives can be optimised simultaneously. In such an approach there will not be a single solution - due to the conflicting nature of the objectives - instead a range of possible trade-off solutions will exist. Without additional information all solutions are equally preferred (Coello, 2006).

When evaluating solutions with respect to multiple objectives we can distinguish between them via the notion of Pareto dominance. The set of trade-off solutions to a multiple objective problem are termed *Pareto optimal solutions*. For each such solution any improvement in one objective will result in a decrease within one or more of the other objectives. This set of solutions map out the trade-off front of the problem termed the *Pareto front*. Solutions can be compared based upon their dominance with respect to the set of objectives. Given 2 individuals I_1 and I_2 : For the set of objectives O , I_1 is said to dominate I_2 if for at least one of the objectives it has a greater value than I_2 and for the other objectives it has an equal to or greater than value compared to I_2 . (Here assuming maximization objectives).

$$O(I_1) \geq O(I_2) \quad (10)$$

I_1 is said to strongly dominate I_2 if for all the objectives it has a greater value than the I_2 (again assuming maximization objectives). This stronger form of dominance is denoted via:

$$O(I_1) > O(I_2) \quad (11)$$

Solutions which are not dominated by any other solutions are termed non-dominated solutions.

INSITE seeks to reduce inconsistency within a set of judgments through modelling inconsistency measures and alteration to the judgments as separate objectives (chosen by the DM) via MOO. Due to the

conflicting nature between objectives of inconsistency and compromise measures, there will not exist a single solution that optimizes all objectives; rather a range of non-dominated solutions will exist. Given a problem with n elements and a complete n by n PCM of judgments from a DM, a Judgment Set of Original judgments O of cardinality J can be selected, containing enough information to reconstruct the whole of the PCM. We seek the set of non-dominated Altered Solutions for the chosen objectives. We represent each Altered solution as a judgment set of cardinality J , denoted as $A = \{a_1, a_2, \dots, a_j\}$ obtained by minimising the set of objectives Λ . The MOO problem can be formulated as:

$$\text{Minimize } [\Lambda] \quad (12)$$

where

$$\Lambda = \{E, B\}$$

The set of objectives Λ consists of two subsets. The first subset E represents one or more measures of compromise objectives chosen by the DM; the second subset B represents one or more measures of inconsistency chosen by the DM. INSITE additionally allows a DM to set constraints both upon the amount of inconsistency reduction they are seeking and upon the amount of compromise they are willing to tolerate in pursuit of reducing inconsistency. Setting constraints upon inconsistency objectives allows a DM to set bounds upon the amount of inconsistency permitted within altered solutions. Thus, a constraint f_j upon an inconsistency objective β_j from objective subset B is defined as:

$$\beta_j(A) \leq f_j \quad (13)$$

For example, when the CR measure is chosen as an objective by a DM, he/she can additionally choose to define a constraint upon the upper value of the objective such as 0.1 (thus following Saaty's recommendation that acceptable PCMs should have a CR value no greater than 0.1) or any other threshold value of the DM's choosing. Setting constraints upon measures of compromise objectives allow a DM to set bounds upon the amount of compromise they are willing to accept to reduce inconsistency. Given a constraint of c_i upon measure of compromise objective ε_j from objective subset E , the following constraint could be defined:

$$\varepsilon_i(A) \leq c_i \quad (14)$$

For example, when the measure of compromise NJR is chosen as an objective by a DM, they could additionally define a constraint upon the objective of 3, in this way seeking only to find Altered Solutions with 3 reversals or less to their original judgments. So, the constrained MOO problem can be formulated as:

$$\text{Minimize } \{E, B\} \quad (15)$$

subject to

$$\begin{aligned} \varepsilon_i(A) &\leq c_i \\ \beta_j(A) &\leq f_j \end{aligned}$$

$$\text{for } i=1,2,\dots,p, \text{ and } j=1,2,\dots,q$$

where E is of size p and B is of size q .

INSITE utilises one or more measures of inconsistency and one or more measures of compromise as objectives. The Consistency Measures Objectives are CR (Saaty, 1977), L, CM (Koczkodaj, 1993), GCI (Aguarón & Moreno-Jiménez, 2003) (for discussion of these see Section 2.1.1) and the Measures of Compromise Objectives are NJV, TJD, STJD, NJR (for definitions and discussions of these see Section 3.2). Furthermore, tackling the problem via a MOO framework allows for additional devised measures of inconsistency to be implemented within INSITE. Therefore, other inconsistency measures could subsequently be implemented and employed within INSITE for utilisation and comparison; similarly, additional measures of compromise that are defined could be straightforwardly added to INSITE. Such additional objective measures would only need to define an evaluation function to be incorporable into INSITE.

3.4 Implementation of a Multi-objective Genetic Algorithm in INSITE

For many real-world multi-objective operational research problems, Evolutionary Computing approaches such as Multi-Objective Genetic Algorithms (MOGA) can be used to swiftly arrive at a high quality approximation of the solution (Gen, Cheng, & Lin, 2008). In INSITE, a MOGA is utilized to seek a set of trade-off solutions, with respect to the set of chosen objectives, which are also as evenly spread along the front as possible. Within INSITE the Multi-Objective Cellular Algorithm (MOCeLL) (Alba et al., 2007) is utilized. In MOCeLL the population is structured into a two-dimensional grid and individuals are only permitted to mate with those individuals close to them in the grid, thus imposing restrictive mating. Offspring that dominate a parent replace the parent in its position in the grid (Alba et al., 2007). MOCeLL utilizes an archive, which retains the best solutions found so far - regarding their Pareto dominance with additional consideration of the even spread of non-dominated solutions across the surface of the trade-off front - and allows the main population to concentrate upon exploring the objective space. An archive has a size parameter, representing the maximum number of solutions the archive can contain, which allows the DM to define the maximum number of solutions to be presented by INSITE. Furthermore, after each generation a defined number of solutions from the archive are added to random positions in the population replacing the individuals in those locations. MOCeLL gives a DM control of the maximum number of solutions that may be returned, all of which will be non-dominated.

Within MOGAs constraints can be considered through various strategies such as discarding infeasible solutions, reducing the fitness of infeasible solutions or repairing infeasible solutions to be feasible; see (Coello, 1999) for a review. Discarding infeasible solutions could result in solutions of high quality Pareto dominance, that are only just infeasible, being lost. Strategies to repair infeasible solutions introduce added complexity regarding defining repair functions. Therefore, in INSITE we implement constraints by reducing the fitness of infeasible solutions to push the population towards the feasible region of the objective space, and in addition by ensuring that only feasible solutions are added to the archive.

INSITE considers any constraints, firstly in the evaluation process, to favour feasible individuals over infeasible solutions and penalise constraint violating solutions. INSITE implements the Constrained Pareto Dominance (Deb, Pratap, Agarwal, & Meyarivan, 2002) as defined for constraint handling in Non-dominated Sorting Genetic Algorithm II (NSGAI). Within Constrained Pareto Dominance feasible solutions are favoured over infeasible solutions pushing the population's individuals towards the feasible area of the Pareto front, see (Deb et al., 2002). INSITE additionally implements a hard constraint upon the archive to allow only feasible solutions to be added to the archive, thus ensuring that only feasible solutions will be presented to the DM. This additionally enhances the feedback operation of MOCeLL as only feasible solutions will be fed back from the archive into the population helping to further steer the population towards the feasible region of the objective space. Use of a MOGA facilitates the finding of a set of trade-off solutions in near-real time. The nature of the surfaces of the trade-off fronts, from the possible objectives utilised within INSITE, allows the MOGA to swiftly find a set of trade-off solutions.

For the examples that follow in the next section, the MOCeLL parameter settings are: population size of 100 (10 x 10 grid); maximum evaluations count of 25,000; selection is performed via binary tournament with single point crossover (with crossover probability 0.9) and bit flip mutation (with probability 0.01) employed. The size of the archive is definable by the DM and stated in each example (with the feedback value set to 25% of the size of the archive). INSITE is independent of a specific prioritization method, so any method can be utilised to derive a ranking from Altered Solutions found. Where preference vectors are calculated in the examples that follow the GM prioritization method (Crawford, 1987) is utilized. In GM a preference vector is derived via the product of each row raised to the inverse power of n . These weights are then usually normalized to sum to 1; see (Crawford, 1987).

4 Numerical Examples

In this section, step-by-step examples of INSITE are presented including comparisons between INSITE and in total four other inconsistency-reducing approaches.

1. Example 1 explores a PCM taken from (Saaty, 1990) and compares INSITE to three other approaches for inconsistency reduction;
2. Example 2 explores a PCM taken from (Saaty, 2000) and compares INSITE to a fourth approach for inconsistency reduction;

3. Example 3 takes a PCM with high levels of both cardinal and ordinal inconsistency and explores how INSITE allows a DM to use different inconsistency measures to suit their preferences;
4. Example 4 shows how a DM can iteratively add constraints to aid in selecting a single solution;
5. Example 5 explores using both multiple inconsistency objectives and using multiple measures of compromise simultaneously.

4.1 Example 1: Comparisons using Saaty's 'buying a house' matrix

Example 1 uses an 8 element PCM, shown in Table 1; the example is taken from (Saaty, 1990), and has been used by Xu and Wei (1999), Cao et al. (2008) and Girsang et al. (2015). The initial CR is 0.17, thus greater than Saaty's 0.1 threshold of acceptance. All three referenced approaches look to derive an altered PCM solution that has a CR value less than 0.1. We compare their solutions against INSITE.

Table 1: Example 1 PCM: [CR: 0.17]

	1	2	3	4	5	6	7	8
1	1	5	3	7	6	6	1/3	1/4
2	1/5	1	1/3	5	3	3	1/5	1/7
3	1/3	3	1	6	3	4	6	1/5
4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8
5	1/6	1/3	1/3	3	1	1/2	1/5	1/6
6	1/6	1/3	1/4	4	2	1	1/5	1/6
7	3	5	1/6	7	5	5	1	1/2
8	4	7	5	8	6	6	2	1

Tackling this problem with INSITE given two DM-chosen objectives of CR and TJD and an archive size of 10, the objective space of solutions found is shown in Figure 2: Left. A dashed vertical line shows the CR threshold of 0.1.

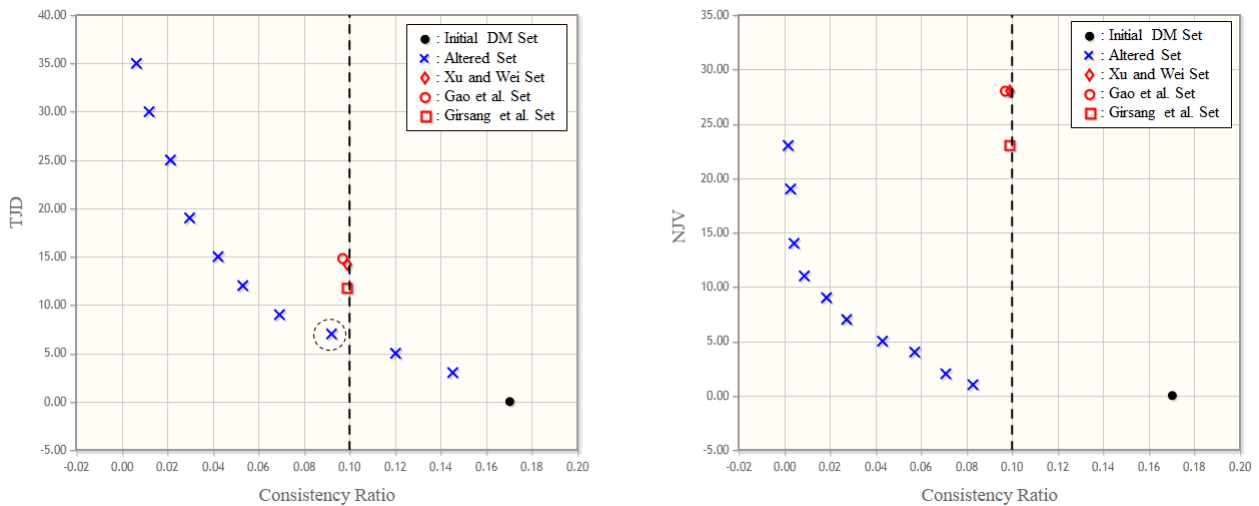


Figure 2: Left: Example 1 Objective Space CR and TJD objectives, Right: Example 1 Objective Space CR and NJV objectives

The DM is then free to review and select any of the 10 solutions found. For instance, the DM could select the first solution along the trade-off front with a CR value less than 0.1, identified via a dotted circle in Figure 2: Left, the PCM of which is shown in Table 2. From this, a DM has a solution with CR less than 0.1 (0.089) and

a meaningful measure of the amount of alteration to reach this – from the TJD value of 7, the DM sees that a total of 7 judgment scale steps of compromise were needed to find this solution.

Table 2: Example 1 possible INSITE solution [CR: 0.089 TJD: 7]

	1	2	3	4	5	6	7	8
1	1	5	3	7	6	6	1	1/4
2	1/5	1	1/3	5	3	3	1/4	1/7
3	1/3	3	1	6	3	4	3	1/5
4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8
5	1/6	1/3	1/3	3	1	1	1/5	1/6
6	1/6	1/3	1/4	4	1	1	1/5	1/6
7	1	4	1/3	7	5	5	1	1/2
8	4	7	5	8	6	6	2	1

Additionally in Figure 2: Left we have plotted the solutions found for this problem from the approaches in (Xu & Wei, 1999), (Cao et al., 2008) and (Girsang et al., 2015), the PCMs of which are shown in Tables 3, 4 and 5 respectively.

Table 3: Example 1 solution from Xu and Wei (Xu & Wei, 1999) [CR: 0.097 TJD: 14.219]

	1	2	3	4	5	6	7	8
1	1	4.52	2.34	7.52	5.89	5.69	0.43	0.29
2	0.22	1	0.33	4.52	2.67	2.58	0.22	0.15
3	0.43	3.07	1	6.75	3.46	4.19	4.16	0.25
4	0.13	0.22	0.15	1	0.37	0.29	0.13	0.10
5	0.17	0.37	0.29	2.68	1	0.56	0.19	0.15
6	0.18	0.39	0.23	3.48	1.78	1	0.20	0.15
7	2.35	4.49	0.24	7.48	5.07	4.90	1	0.50
8	3.42	6.79	4.02	9.62	6.78	6.55	1.99	1

Table 4: Example 1 solution from Cao et al (Cao et al., 2008) [CR: 0.099 TJD:15.63]

	1	2	3	4	5	6	7	8
1	1	4.44	2.37	7.67	5.86	5.61	0.42	0.30
2	0.23	1	0.32	4.42	2.62	2.54	0.23	0.15
3	0.42	3.12	1	6.92	3.54	4.28	4.51	0.25
4	0.13	0.23	0.15	1	0.38	0.29	0.13	0.10
5	0.17	0.38	0.28	2.63	1	0.57	0.19	0.14
6	0.18	0.39	0.23	3.42	1.75	1	0.21	0.15
7	2.38	4.41	0.23	7.61	5.10	4.87	1	0.50
8	3.37	6.73	4.02	9.71	6.92	6.70	1.99	1

Figure 2: Left shows that the three solutions found via the referenced approaches are dominated with respect to the trade-off front mapped out by the solutions found by INSITE with regards to CR and the total amount of deviation undergone by the original judgments. Additionally, from the three referenced approaches solutions it will be more difficult for a DM to discern how their judgments have changed as the solutions contain values outside the originally used scale. Although solutions found via Girsang et al. find judgments within the bounded range of the utilized scale the judgments fall between the scale steps making interpretability difficult by a DM, regarding how their judgments have changed.

Table 5: Example 1 solution from (Girsang et al., 2015) [CR: 0.099 TJD: 11.71]

	1	2	3	4	5	6	7	8
1	1	4.20	2.40	7.40	5.80	5.40	0.42	0.29
2	0.24	1	0.33	4.40	2.60	2.40	0.25	0.15
3	0.42	3.00	1	6.80	3.40	4.20	4.40	0.24
4	0.14	0.23	0.15	1	0.36	0.29	0.14	0.11
5	0.17	0.38	0.29	2.80	1	0.56	0.20	0.15
6	0.19	0.42	0.24	3.40	1.80	1	0.20	0.15
7	2.40	4.00	0.23	7.00	5.00	5.00	1	0.50
8	3.40	6.80	4.20	9.00	6.80	6.80	2.00	1

A total deviation measure is calculated for these solutions based upon the amount of scale steps that each modified judgment has undergone, the sum of which is used to plot these solutions within the objective space in Figure 2: Left. For example, taking the judgment between elements 3 and 7, for the solution in (Xu & Wei, 1999) in Tables 1 and 3, the judgment of 6 has changed to a judgment of 4.155, which in terms of deviation scale steps we calculate as 1.845 steps. Similarly, for fractional judgments we determine the deviation again as the amount of scale steps that occur. For example, taking the judgment between elements 3 and 8, for the solution in (Xu & Wei, 1999) from Tables 1 and 3 the judgment of 1/5 (0.2) has altered to 0.249 which in terms of deviation scale steps represents 0.976 steps. We calculate the total deviation of these three referenced approaches' solutions as 14.219 for Xu & Wei, 15.63 for Cao et al. and 11.71 for Girsang et al.; all three give greater amounts of total deviation than the selected solution found by INSITE in Table 2. Here INSITE has sought a set of trade-off solutions with respect to the pair of user-chosen objectives of CR and TJD, therefore its concern was to find trade-off solutions where preservation of the original judgments was measured with respect to the total judgement deviation.

Further analysis of the results can be performed regarding the number of judgments that require change to find these solutions; INSITE is able to find solutions will small numbers of the initial judgments changing: in the solution found in Table 2 only four of the original judgments have been altered. Conversely, in the solutions found via the other referenced approaches large numbers of judgment changes occur: 28 for Xu & Wei and Cao et al., and 23 for Girsang et al. respectively. Such large numbers of original judgment changes again make it harder for a DM to analyse an altered set of judgments and discern how their judgments have been altered to achieve reduced inconsistency. This analysis is of a trade-off solution that has been found by INSTIE with respect to the pair of user-chosen objectives of CR and TJD (therefore not from explicitly looking to optimise with respect to the number of judgments that have changed).

However, the use of the measures of compromise by INSITE gives a DM flexibility regarding how alteration to their judgments is measured allowing the user to define what preservation of the original judgment information is desirable for them. For example, given a scenario where a DM seeks solutions with low inconsistency whilst also looking to specifically minimise the number of judgments that change. By utilizing objectives of CR, and this time NJV instead of TJD for the above problem, INSITE finds a set of trade-off solutions, shown in Figure 2: Right along with the plotted solutions of the referenced approaches. Here we observe that INSITE finds a trade-off front of solutions with at one edge a solution of the most amount of inconsistency reduction possible from a single judgment change, and at the other edge a solution of the minimum number of changed judgment required to remove all inconsistency. The DM is then free to review and select any of the 10 solutions found. For instance, the DM could select the first solution along the trade-off front with a CR value less than 0.1, from this, the DM has a solution with CR less than 0.1 (0.083) and for which only a single judgment from the original set has been altered.

4.2 Example 2: Comparison using Saaty's 'school selection' matrix

Example 2 uses a 6 element PCM, shown in Table 6, taken from (Saaty, 2000), and has been used by Zhang et al. (2014). The initial CR is 0.23, thus again greater than Saaty's 0.1 threshold of acceptance. Zhang et al. look to find a single solution with reduced inconsistency. We compare their solution against INSITE.

Table 6: Example 2 PCM [CR: 0.23]

	1	2	3	4	5	6
1	1	4	3	1	3	4
2	1/4	1	7	3	1/5	1
3	1/3	1/7	1	1/5	1/5	1/6
4	1	1/3	5	1	1	1/3
5	1/3	5	5	1	1	3
6	1/4	1	6	3	1/3	1

By tackling this problem with INSITE given two DM-chosen objectives of CR and TJD and an archive size of 10, the objective space of solutions found is shown in Figure 3: Left. A dashed vertical line shows the CR threshold of 0.1. In this objective space we have also plotted the solution from Zhang et al., which has CR 0.03 and TJD 20.64; the PCM from their solution is shown in Table 7. From the set of trade-off solutions found from INSITE, the DM is able to glean information about the problem and the trade-offs involved in reducing inconsistency within their judgments, which is not possible when seeking only a single solution. From Figure 3: Left we observe, as previously, that INSITE is able to map out a trade-off front of solutions that dominates the solution found from the compared approach, with regards to CR and the total amount of deviation undergone by the original judgments. Furthermore, the judgments found for the Zhang et al. solution fall outside of the original scale utilized making it harder for a DM to discern how their judgments have changed.

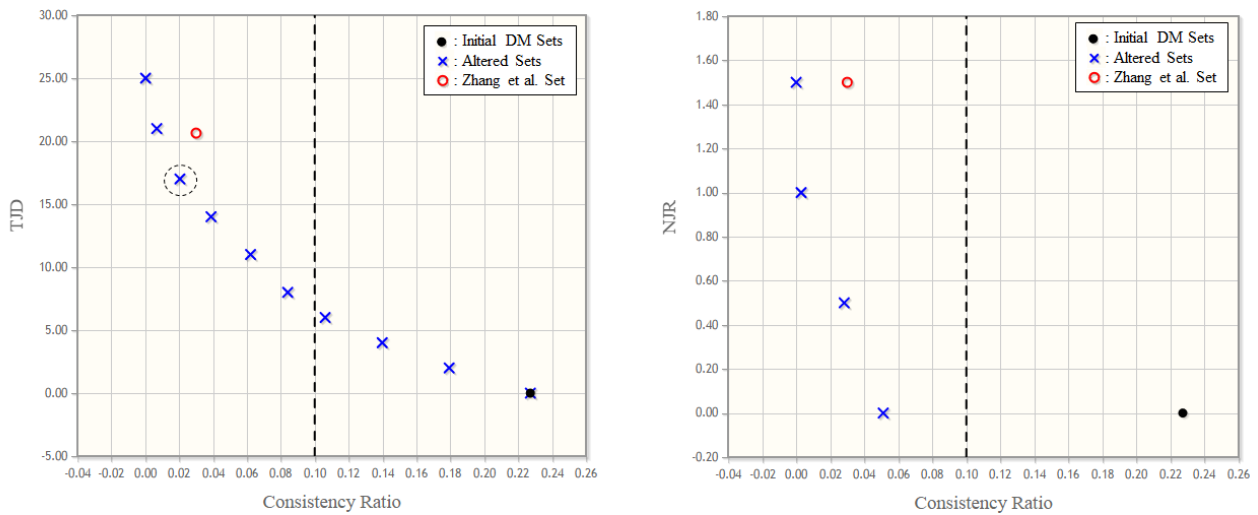


Figure 3: Left: Example 2 Objective Space CR and TJD objectives, Right: Example 2 Objective Space CR and NJR objectives

Table 7: Example 2 solution from (Zhang et al., 2014) [CR: 0.03 TJD: 20.64]

	1	2	3	4	5	6
1	1.00	2.26	4.58	1.95	1.31	2.69
2	0.44	1.00	4.96	2.34	0.34	1.19
3	0.22	0.20	1.00	0.35	0.15	0.26
4	0.51	0.43	2.87	1.00	0.43	0.51
5	0.76	2.93	6.66	2.32	1.00	2.84
6	0.37	0.84	3.91	1.96	0.35	1.00

The DM can review and select any of the 10 solutions found by INSITE. The use of a measure of compromise, to measure alteration, enables the DM to compare the solutions found via relevant, and meaningful measures regarding the amount of alteration needed to reach the CR value of each solution. For example, the DM could select the solution identified in Figure 3: Left via a dotted circle, the PCM of which is shown in Table 8. From

this, the DM has a solution that dominates the solution found via Zhang et al. with a smaller CR value and a smaller TJD value. Here the INSITE solution has CR of 0.02 and TJD 17.

Table 8: Example 2 possible INSITE solution [CR: 0.02 TJD: 17]

	1	2	3	4	5	6
1	1	4	8	2	3	4
2	1/4	1	7	1	1	1
3	1/8	1/7	1	1/5	1/5	1/5
4	1/2	1	5	1	1	1
5	1/3	1	5	1	1	1
6	1/4	1	5	1	1	1

INSITE gives a DM flexibility regarding how alteration to their judgments is measured enabling a use scenario where a DM seeks solutions with low CR whilst also looking to minimise the number of judgment reversals, through utilizing NJR as an objective instead of TJD. For the above problem, utilising the objectives of CR and NJR INSITE finds a set of trade-off solutions, shown in Figure 3: Right, along with the solution found from the Zhang et al. approach. Here we observe that INSITE finds a trade-off front of solutions with at one edge a solution of the most amount of inconsistency reduction possible without any judgment reversals, and at the other edge a solution of the minimum number of reversals required to remove all inconsistency. Furthermore, we observe that the trade-off front mapped out by the solutions found via INSITE dominates the Zhang solution, with respect to the DM’s chosen needs, here of CR and the number of judgment reversals.

4.3 Example 3: Inconsistency measures

Next, we present an example with high initial levels of cardinal and ordinal inconsistency to illustrate how different inconsistency measures of a DM’s choosing can be utilised within INSITE. A PCM for a 9 element problem is shown in Table 9, the initial inconsistency measure are CR: 0.76, L: 9, CM: 0.99 and GCI: 50.8. To seek inconsistency reduction via INSITE a DM can choose an inconsistency measure of their preference. For example, if the DM seeks to reduce cardinal inconsistency and wishes to utilise the CR measure they can select CR as an objective. If further the DM chooses STJD as a measure of compromise objective and an archive of size 10, the solution space is shown in Figure 4: Left. From this, a DM can see that a large amount of alteration is required to find a solution below the CR threshold of 0.1.

Table 9: Example 3 PCM

	1	2	3	4	5	6	7	8	9
1	1	1/8	1/3	1/7	1/3	1	8	1/9	1/4
2	8	1	5	1/2	1/3	4	3	7	5
3	3	1/5	1	2	1/2	1/6	7	7	1/9
4	7	2	1/2	1	1	5	2	2	1/9
5	3	3	2	1	1	7	6	5	6
6	1	1/4	6	1/5	1/7	1	2	1/6	1
7	1/8	1/3	1/7	1/2	1/6	1/2	1	1	8
8	9	1/7	1/7	1/2	1/5	6	1	1	1/8
9	4	1/5	9	9	1/6	1	1/8	8	1

Alternatively, if a DM seeks to reduce the number of cycles within their judgments, being more interested in ordinal inconsistency reduction, then they can instead choose L as an inconsistency objective. Given again that the DM chooses STJD as a measure of compromise objective and an archive size of 10, the solution space found is shown in Figure 4: Right. From this, we see the shape of the objective space across the range of values of L for the minimal amount of STJD. We observe a large jump in the amount of alteration to reduce the number of cycles from 4 to 3 and again to reduce the number of cycles from 1 to 0. Such analysis of the

objective space should help a DM to better understand the nature of inconsistency reduction for their judgments and make a more informed choice regarding reducing inconsistency.

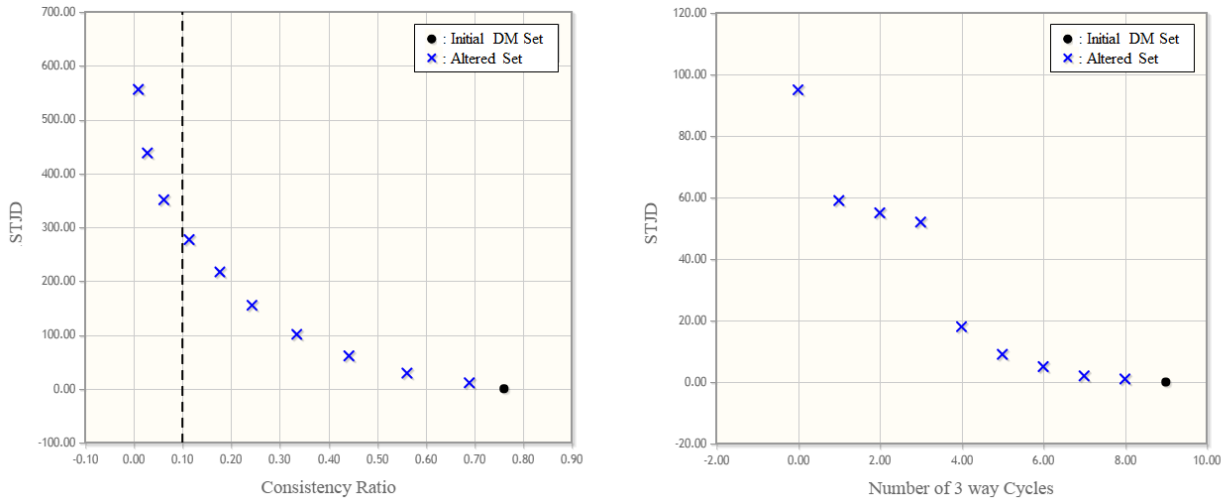


Figure 4: Example 3, Left: CR and STJD objectives. Right: L and STJD objectives

Conversely, if a DM is concerned with reducing inconsistency by looking to reduce the largest inconsistent judgment triple then the CM can instead be chosen as an inconsistency objective. Given that the DM again chooses STJD as a measure of compromise objective and an archive size of 10, the solution space found is shown in Figure 5: Left. From this objective space we observe the convex nature of the front for this objective pair. We further observe that there is little reduction in CM towards the edge of the trade-off front near the initial judgments yet, at the other edge of the front larger decreases in CM are achieved for lower amounts of additional compromise.

If a DM instead wishes to utilise a distance-based inconsistency measure of the distance between judgments and a derived preference vector, then the GCI can be chosen as the inconsistency measure. The solution space with STJD as a measure of compromise objective and an archive size of 10 is shown in Figure 5: Right. Additionally plotted as a vertical dotted line is the GCI threshold measure (Aguarón & Moreno-Jiménez, 2003) (0.37 when $n > 5$). From this plot and Figure 4: Left the strong relationship between CR and GCI (Aguarón & Moreno-Jiménez, 2003) is clearly conveyed visually to the DM.

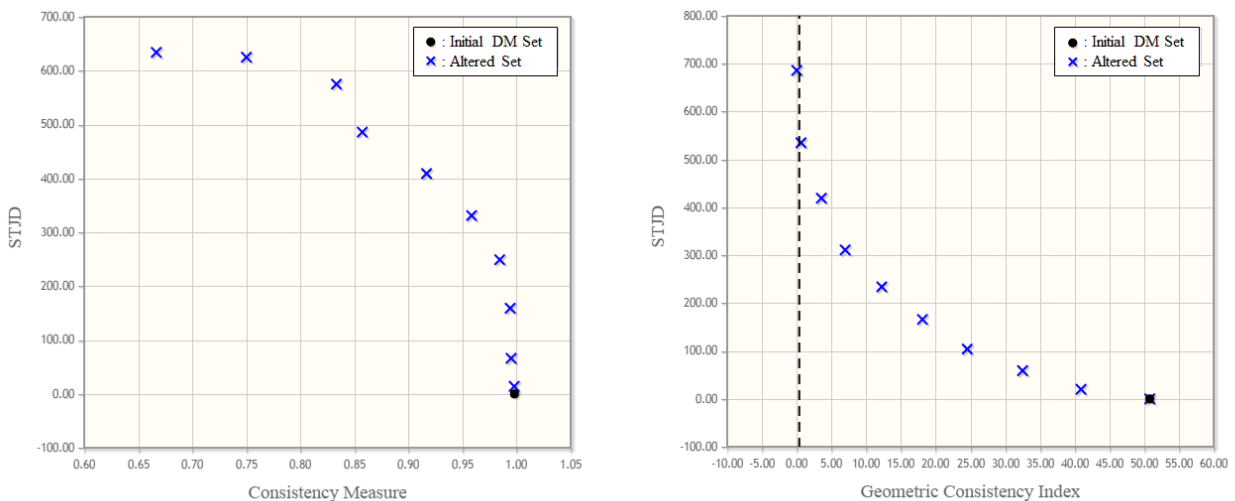


Figure 5: Example 3, Left: CM and STJD objectives. Right: GCI and STJD objectives

This example shows how INSITE is versatile to the preferences of a DM regarding how inconsistency will be defined and measured in their judgments. Furthermore, we see how INSITE allows for knowledge of the nature of the trade-offs involved in the problem to be revealed to the DM, thus giving them a richer understanding of both the process and the outcomes.

4.4 Example 4: Interactively adding constraints

This example explores how interactive analysis of an objective space through iteratively adding constraints can aid a DM in the selection of a single solution. DM judgments for a 5-element problem and a preference vector derived using the GM prioritization method are shown in Table 10 along with the initial CR inconsistency measure. Then, given a DM chooses objectives of TJD and CR and an archive size of 10, the initial objective space is shown in Figure 6.

Table 10: Example 4 PCM [CR: 1.08]

	1	2	3	4	5	w
1	1	8	6	1/2	1/5	0.25
2	1/8	1	7	2	7	0.30
3	1/6	1/7	1	5	1/2	0.10
4	2	1/2	1/5	1	1/2	0.12
5	5	1/7	2	2	1	0.23

From this the DM can get an overview of the objective space and the nature of the trade-offs between the objectives over the front for the problem, helping them to then add feasible constraints. For example, the DM may conjecture that it is feasible to seek a solution whose CR value is less than 0.1, so he/she sets a constraint upon CR to only find solutions with CR of 0.1 or less. Additionally, to focus upon this area, they may increase the archive size to a maximum of 20 and perform the search with these new parameters and constraint. The objective space with this added constraint is shown in Figure 7: Left with the constraint edge shown as a dotted red line. From this constrained objective space, the DM might further conjecture that the amount of deviation increase past 20 then yields little further reduction in inconsistency. Therefore, they may decide to add an additional constraint upon the upper amount of deviation to be less than 20 (so 19 or less). The new constrained objective space with this additional constraint added is shown in Figure 7: Right. From this second constrained objective space the DM can observe there are only two solutions, with TJD values 18 and 19 respectively. The DM can then analyse these solutions (Table 11), along with the values of the objectives and their preference vector rankings of the elements (again derived using the GM prioritization method).

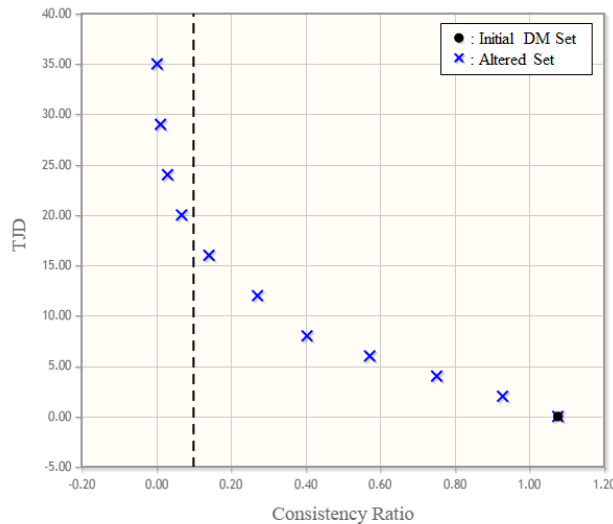


Figure 6: Example 4 Objective Space. CR and TJD objectives

By analysing the visualization of the objective space, the DM can clearly see the total deviation compromise needed to achieve these solutions, and that both have attained over 90% reduction in initial CR inconsistency. Regarding the ordinal rankings of the preference vectors compared to the initial judgments preference vector, only 1 change has occurred - between elements 3 and 4 - for both solutions. From such analysis, the DM may conjecture that for the TJD 19 solution the additional reduction in inconsistency is worth the additional deviation step. Therefore, the solution with inconsistency of CR 0.058 and TJD 19 is chosen.

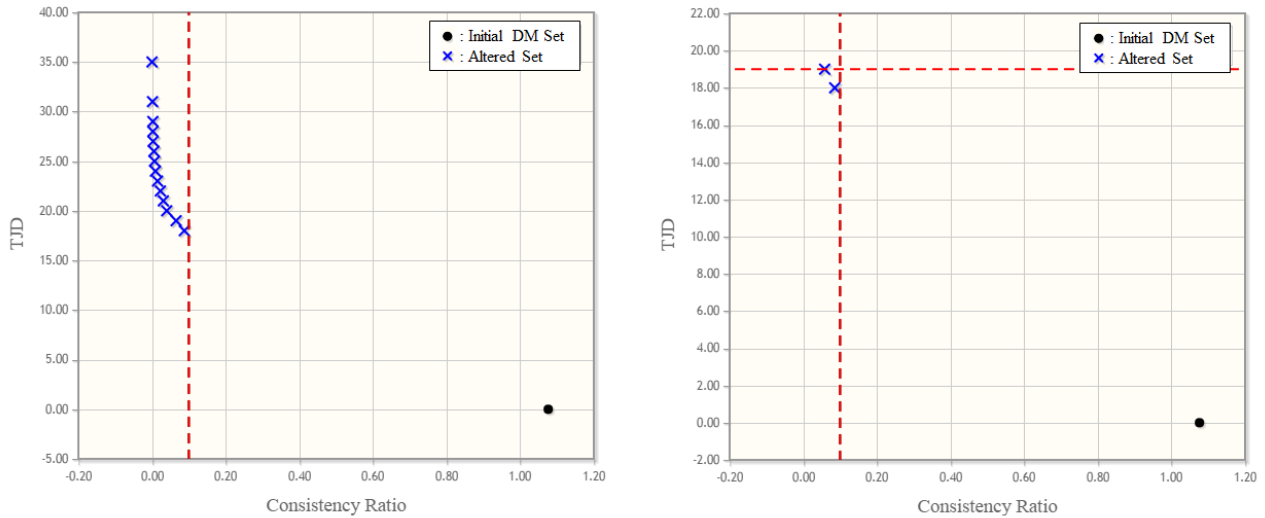


Figure 7: Example 4, Left: Objective Space with CR constraint, Right: Objective space with CR and TJD constraints

Table 11: Example 4 Constrained objective space solutions

TJD:18 CR:0.085							TJD:19 CR:0.058						
	1	2	3	4	5	w		1	2	3	4	5	w
1	1	1	6	1	1	0.24	1	1	1	6	1	2	0.27
2	1	1	7	2	7	0.41	2	1	1	7	2	7	0.40
3	1/6	1/7	1	1/2	1/2	0.06	3	1/6	1/7	1	1/2	1/2	0.06
4	1	2	2	1	1	0.16	4	1	1/2	2	1	1	0.16
5	1	1/7	2	1	1	0.13	5	1/2	1/7	2	1	1	0.11

4.5 Example 5: Larger objective sets

INSITE is not constrained to objective sets of size 2; in this example we illustrate how a DM can choose larger objective sets using the judgments from Example 4 shown in Table 10. A DM could utilise multiple measures of compromise simultaneously, for example, they could choose to use 3 objectives of CR, TJD and NJV. From this objective set, INSITE will look for Altered Solutions with low CR values, with minimum deviation and look to minimise the number of judgments that change. The 3-dimensional objective space for this objective set with a large archive size defined to help emphasise the nature of the front is shown with respect to CR and NJV in Figure 8: Left. From this, we observe a pattern within the objective space of multiple solutions with the same NJV value but various levels of CR; these solutions represent different levels of deviation for the same level of NJV. We see also that as CR tends towards 0 the range of CR values for solutions with the same NJV value decreases. A DM could additionally perform analysis between the measures of compromise; for example, the same 3-dimensional objective space shown with respect to TJD and NJV is shown in Figure 8: Right. From this, a DM can see clearly the relationship between the measures of compromise for this set of judgments. We see a positive correlation relationship between the measures yet for higher amounts of compromise this relationship appears to weaken. Additionally, for such an objective set a DM could add multiple constraints to for example, define thresholds on both TJD and NJV.

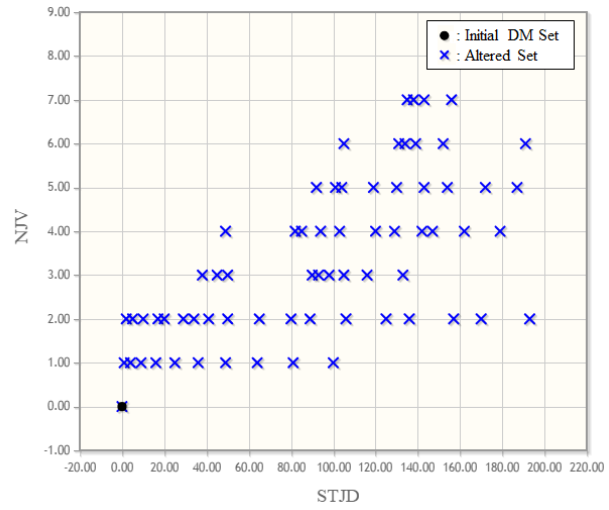
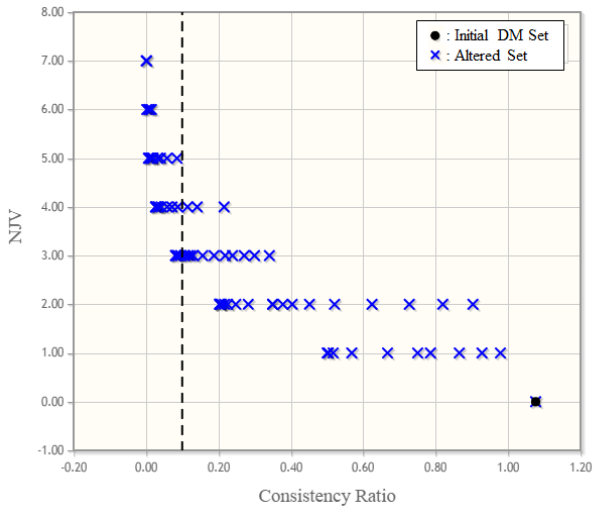


Figure 8: Example 5 Multiple measures of compromise objectives analysis

A DM could also utilise multiple measures of inconsistency simultaneously. For example, say a DM chooses three objectives of, the TJD measure of compromise, and CR and L inconsistency measures. Here INSITE will look to reduce both cardinal and ordinal inconsistency simultaneously. The 3-dimensional objective space is shown with respect to L and TJD in Figure 9: Left. From this, we see a number of solutions with an L value of zero - which have all ordinal inconsistency removed but with a range of different deviation values. Additionally, a DM could analyse the relationship between the inconsistency measures for this objective space as shown with respect to CR and L in Figure 9: Right. In this view, we see how the objectives both converge to 0 inconsistency solutions at one edge of the objective space. However, we also observe the outline of two arcs of solutions from the initial judgment set edge of the front to a solution with all inconsistency removed, highlighted as dotted red arcs on the plot. This demonstrates the different emphases of the objectives of cardinal and ordinal inconsistency and the importance of flexibility to allow a DM to decide upon how inconsistency reduction will be measured to best suit their needs.

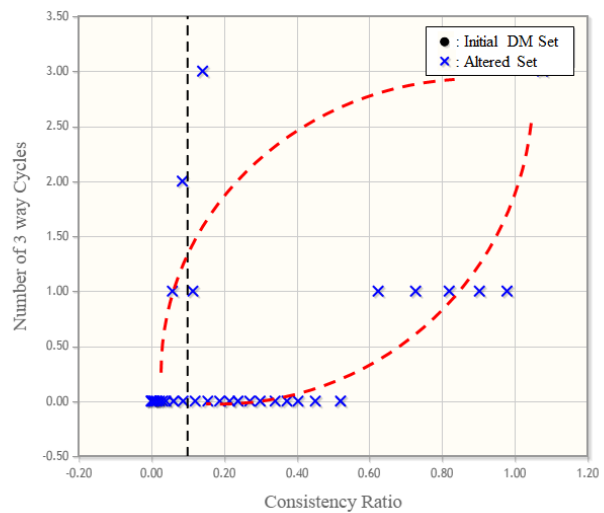
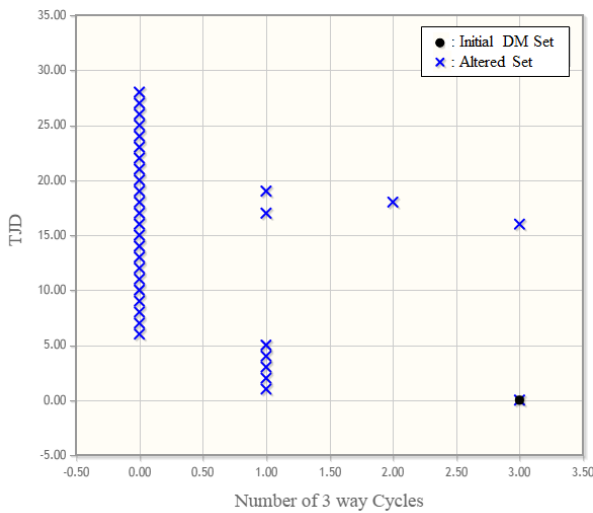


Figure 9: Example 5 Multiple inconsistency measure objectives analysis

5 Conclusions

This paper has presented INSITE, an approach to reducing inconsistency within a set of PC judgments via MOO. INSITE seeks to optimally reduce inconsistency within a set of judgments, by modelling inconsistency reduction and alteration to the judgments as separate objectives, to find a set of trade-off solutions between the conflicting objectives. From this, a DM can glean knowledge of the trade-offs involved between inconsistency reduction and judgment alteration, helping facilitate an evidential, transparent, auditable and traceable process. Judgments are altered in such a way that they maintain the original scale utilised by a DM, thus allowing

him/her to more easily discern how their judgments have altered. A DM has control over the type of inconsistency reduction to seek, as both cardinal and/or ordinal inconsistency measures can be chosen as objectives. A DM also has control over how alteration is measured to meet their needs via the measures of compromise. Finally, a DM is able to set their own constraints relating to the amount of inconsistency reduction they are seeking to achieve and/or the amount of alteration they are willing to tolerate.

Future work will investigate further traceability within MOO by investigating group aggregation of multiple DMs' views, where invariably compromise is needed between different DMs' views to reach consensus. Moreover, future work will look to develop the INSITE approach into a hosted online tool so as to be freely available as a web-based decision support tool.

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